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The Complete Monotonicity of the Rayleigh Function

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1. INTRODUCTION

In the present note we discuss and establish a real-analytic property satisfied by the series,

$$\sigma_n(u) = \sum_{k=1}^{\infty} j_{u,k}^{-2n}, \qquad n \geqslant 1,$$

where $\{j_{u,k}\}$ is the infinite sequence of the zeros of the integral function, $z^{-u}J_u(z)$ which lie in the right half plane. These zeros are ordered by $|\operatorname{Re} j_{u,k}| < |\operatorname{Re} j_{u,k+1}|$. The above symmetric functions of the zeros of the Bessel functions are important and useful in many ways, particularly as the coefficients of the odd meromorphic function, $(J_{u+1}/2J_u)(z)$, viz., $\sum_{n=1}^{\infty} \sigma_n(u)\, z^{2n-1} = J_{u+1}(z)/2J_u(z) \equiv w_u(z)$, which is a series solution to the Riccati-type equation, $w' + (1+2u)(w/z) - 2w^2 = \frac{1}{2}$. (cf. [1, 1a, 2].) The basic and other properties of these coefficients can be found elsewhere (see the references). Lord Rayleigh is often credited with the first application of these functions [4]. $\sigma_n(u)$ is called the Rayleigh function of order n, in u. Here we present them as an interesting collection of unusual examples of completely monotonic functions; thus the coefficients of $w_u(z)$ are Laplace–Stieltjes integrals on a semi-axis (c, ∞).

2. COMPLETE MONOTONICITY

On intervals $X = R_c \equiv (c, \infty)$ or $\bar{R}_c \equiv [c, \infty)$, a nonnegative real function, f, in $C^\infty(X)$ is said to be completely monotonic (and we write cm) on X if $(-1)^m f^{(m)}(x) \geqslant 0$, $\forall m \geqslant 0$. For such functions successive derivatives alternate in sign, beginning with $f'(x)$ which is negative or zero, on X. For $I \subseteq R$, a $C^\infty(I)$-function, f, is said to be absolutely monotonic (and we write am) if $f^{(m)}(x) \geqslant 0$, $\forall m \geqslant 0$. Similar definitions hold for finite intervals. We observe that finite sums and products, as well as multiplications by

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nonnegative constants, of cm (or am) functions are still cm (am). An absolutely or completely monotonic function is always real-analytic in its region of monotonicity (Bernstein's theorem); to this region, I, in fact, f is the restriction of a cortain function analytic in a region of the complex plane that contains I. The cm and am functions have a considerable literature all their own, and the reader may consult [5, 6] and their references for extra details.

THEOREM. *The Rayleigh functions, $\sigma_n(u)$, as well as $(u+n)\,\sigma_n(u)$ and $(1+u)^n\,\sigma_n(u)$, are cm functions of u on R_{-1}, while $-\sigma_n(u)$ is am on $(-\infty, -n)$; consequently, the coefficients of $\frac{1}{2}(J_{u+1}/J_u)(z)$ are Laplace–Stieltjes integrals*:

$$\sigma_n(u) = \int_0^{\infty} e^{-ut}\, dg_n(t), \qquad -1 < u < \infty, \tag{1}$$

where $g_n(t)$ is a certain non-decreasing function on $\bar{R}_0$ or of bounded variation on finite subintervals of $\bar{R}_0$.

Proof. For $n = 1$, $\sigma_1(u)$ is the function, $1/(4(u+1))$, which, we see, is cm on R_{-1}. When $n > 1$, it is most convenient to consider the recurrence equation,

$$\sigma_n(u) = (u+n)^{-1} \sum_{k=1}^{n-1} \sigma_k(u)\,\sigma_{n-k}(u), \tag{2}$$

derived from the series $\frac{1}{2}\,(J_{u+1}/J_u)(z) = \sum_{n=1}^{\infty} \sigma_n(u)\, z^{2n-1}$ in [1, (22)]. The factor $(u+n)^{-1}$ is clearly cm on R_{-1}. If we assume now that $\sigma_k(u)$ is cm on R_{-1} for all $k < n$, and recall our earlier observation that the cm property is preserved by finite addition and multiplication, the cm property of $\sigma_n(u)$ on R_{-1} follows by induction from (2). Hence $\sigma_n(u)$ is cm, $\forall n \geqslant 1$, on R_{-1}. This granted, then a multiplication of both sides of (2) by $(u+n)$ shows, too, that $(u+n)\,\sigma_n(u)$ is cm on R_{-1}. Again, when $n > 1$, let us rewrite $(1+u)^n\,\sigma_n(u)$, using (2) into the form,

$$(1+u)^n\,\sigma_n(u) = (u+n)^{-1} \sum_{k=1}^{n-1} \{(1+u)^k\,\sigma_k(u)\}\{(1+u)^{n-k}\,\sigma_{n-k}(u)\}, \tag{2a}$$

and write $-\sigma_n(u)$ as,

$$-\sigma_n(u) = -(u+n)^{-1} \sum_{k=1}^{n-1} (-\sigma_k(u))(-\sigma_{n-k}(u)). \tag{2b}$$

Then another induction, with (2a) and (2b), shows, respectively, that $(1+u)^n\,\sigma_n(u)$ is cm on R_{-1} and $-\sigma_n(u)$ is am on $(-\infty, -n)$.

However, by a theorem of Widder [6a], a necessary and sufficient condition for a function, f, to be cm on R_c is that $f(x) = \int_0^\infty \exp(-xt)\, dg(t)$, where $g(t)$ is a non-decreasing function on $\bar{R}_0$ such that the integral converges $\forall x \in R_c$. Furthermore, a function, f, is of the form, $\int_0^\infty \exp(-xt)\, dg(t)$, where $g(t)$ is of bounded variation on any finite subinterval of $\bar{R}_0$, and is such that the integral converges for all x in R_c) iff $f(x)$ is the difference of two cm functions on R_c. Now since $\sigma_n(u)$ is cm on R_{-1}, the integral form (1) follows.

3. Remarks

The reader may verify that

$$\frac{1}{2w_u(z)} = 2(u+1)\, z^{-1} - \sum_{n=1}^{\infty} 2\sigma_n(u+1)\, z^{2n-1};$$

a part of the theorem then implies that the coefficients of the analytic part of $(J_u/J_{u+1})(z)$ are *absolutely* monotonic on $u < -n-1$, but that those of the analytic part of $-(J_u/J_{u+1})(z)$ are *completely* so on $(-2, \infty)$. That is, the coefficients of the analytic part of $-1/2w_u(z)$ have Laplace–Stieltjes integral representations in the semi-axis $-2 < u < \infty$.

Arising from the theorem, another interesting (and useful) sequence of cm functions is induced by fixing the argument, $z = \alpha$, and expanding $F(x) \equiv (J_{x+1}/J_x)(\alpha)$ as a function of the order, x, in the interior of the nonnegative real axis. Fix α in $0 \leqslant \alpha < 2$; and for each $k \geqslant 0$, consider (the function of u defined by),

$$c_k(u) \equiv \frac{2}{k!} \sum_{n=1}^{\infty} \alpha^{2n-1} \sigma_n^{(k)}(u), \qquad u > 0.$$

We show $c_k(u)$ exists and c_{2k} is cm on R_0. On $\bar{R}_0$, $0 < \sigma_n(u) \leqslant 4^{-n}$ (use (2) and induction), and since $\alpha \in [0, 2]$, we find that the series,

$$f(u) = \sum_{n=1}^{\infty} \alpha^{2n-1} \sigma_n(u)$$

is uniformly convergent on $\bar{R}_0$; hence we can differentiate term-by-term to get $(2/k!) f^{(k)}(u) = c_k(u)$, on $\bar{R}_0$. (Hence c_k exists.) But since the $(2k)$th derivative of a cm function is also cm, the theorem implies that c_{2k} is cm on $(0, \infty)$, but that the odd-induxed c_k's do not have this property. We will not formally show here, but we must remark, however, that $c_k(u)$ is the kth coefficient of $F(x) = (J_{x+1}/J_x)(\alpha)$ about $x = u > 0$. All the even-indexed ones, we have seen, are cm functions of the centers of expansion. Interestingly, no inspection or representation formulas of the higher derivatives of $\sum_{s=1}^{\infty} j_{u,s}^{-2n}$, with respect to u, are needed for the purposes of checking the cm–am properties in the respective regions.

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